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OPTIMAL SYNTHESIS IN THE PROBLEM OF IMPULSIVE CORRECTION OF MOTION*

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A minimax problem of correcting the motion of a dynamic system acted upon by perturbing forces of restricted magnitude is considered. The corrective action is carried out in the form of impulsive control, with a restriction on the total magnitude of the impulses and their number. This formulation models the problem of the impulsive correction of motion of an aircraft acted upon by external perturbations. The problem represents, in fact, a differentially-impulsive game /1/ in which the player controlling the correction aims to secure for himself a guaranteed minimum of the terminal functional. Following the methods used in /2/ we construct, for the problem with isotropic dynamics, an optimal synthesis of the correction instances. The present paper touches on the work done in /3-5/.

Let us consider a controlled dynamic system whose motion in the time interval $[t_0, T]$ is given by a differential vector equation with initial conditions

$$x^{\cdot} = y,\ y^{\cdot} = u + v,\ x(t_0) = x^\circ,\quad y(t_0) = y^\circ \tag{1}$$

Here x, y, u, v are vectors of the same dimensions. The controls $u(t)$ are realized impulsively, while the samples of the interference $v(t)$ are assumed to be differentiable and to satisfy the following constraints:

$$u(t) = \sum_{k=1}^{N} u_k \delta(t - t_k),\quad t_0 \leqslant t_1 < \ldots < t_N < t_{N+1} = T,\quad |v(t)| \leqslant 1 \tag{2}$$

where $\delta(t)$ is the delta function and $t_1, \ldots, t_N$ are the instants of correction chosen either in advance, or during the motion. The quantity Q characterizes the sum of the possibilities of correction. We assume that the correction should ensure the minimum value of the quantity

$$J = |x(T)| \tag{3}$$

In the present paper we use the minimax (game theoretic) approach, and the corrective action $u(t)$ is constructed under the assumption of the worst sample of the interference $v(t)$. This method of corrective action is found from the condition $\min_u \max_v J$. Here the minimization is carried out over the realization of the noise $v(t)$ belonging to the class of piecewise continuous functions observing the constraint of (2), and the minimization is carried out over the class of admissible synthesizing (positional) controls.

Let us describe a class of admissible positional controls u. The current state of the system is described by the position (z', q, t, k), where $z' = (x, y)$, $t \in [t_0, T]$, $k = 0, 1, \ldots, N$ and q is the total intensity summed over k allowed impulses satisfying the constraint $q \in [0, Q]$. The positional control (synthesis) is specified in the (z', q, t) space by the signal surfaces Γ_k separating the region in question of this space for every $k = 1, \ldots, N$ into two sets G_k and D_k, and by the functions $u_k = u_k(z', q, t)$, $|u_k(z', q, t)| \leqslant q$. In the region G_k we have, by definition, $u_k(z', q, t) = 0$. The surface Γ_k belongs to the boundary of the closed set D_k. The set (u_k, Γ_k), $k = 1, \ldots, N$ is the admissible synthesis and we shall denote it by u.

Let us define the Bellman function $S_k(z', q, t)$, $k = 0, 1, \ldots, N$, $z \in E^n$, $q \in [0, Q]$, $t \in [t_0, T]$ by the relations

$$S_k(z', q, t) = \min_u \sup_v J,\ S_0(z', q, t) = J \tag{4}$$

under the assumption that a minimum exists. The extrema in (4) are calculated over those parts of admissible syntheses and interference, which determine the motion from the position (z', q, t, k).

Let us make the change of variables, $z(t) = x(t) + (T - t)y(t)$. Introducing a new independent variable t' by means of the relations $T' - t' = (T - t)^2/2$, $T' = T^2/2$ and determining the quantities u_k', Q' from the formulas $u_k' = \sqrt{2}u_k$, $k = 1, \ldots, N$, $Q' = \sqrt{2}Q$, we find that problem (1)-(3) can be written in the form

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$$dz/dt = (T-t)^{1/2}u + v, \quad 0 \leqslant t \leqslant T, \quad |v| \leqslant 1$$
$$u(t) = \sum_{k=1}^{N} u_k \delta(t-t_k), \quad \sum_{k=1}^{N} |u_k| \leqslant Q, \quad J = |z(T)| \tag{5}$$

In deriving relations (5) we have taken into account the relation $(2(T'-t_i'))^{1/2}\delta(t'-t_i') = \delta(t-t_i)$. Here and henceforth we shall omit, for simplicity, the primes on the quantities T, t, t_k, u_k.

It was shown in /2/ that the Bellman function in problem (5) can be written in the form

$$S_k(|z|, q, \tau) = |z|\,\varphi_k(\xi, \eta), \quad \tau = T-t, \quad \xi = \tau/|z|$$
$$\eta = q|z|^{-1/2}, \quad \varphi_k(\xi, \eta) = (1+\xi)F_k(\eta(1+\xi)^{-1/2}), \quad k = 0, \ldots, N \tag{6}$$

where F_k is a function of a single variable. The boundaries Γ_k and the functions F_k are connected by recurrence relations. The values of the functions F_k for known F_{k-1}, Γ_k are given by the formulas

$$F_k(\eta(1+\xi)^{-1/2}) = \xi(1+\xi)^{-1}F_{k-1}((\rho-1)/\xi)$$
$$(\xi, \eta) \in \Gamma_k, \quad k = 1, \ldots, N, \quad \rho = \xi^{1/2}\eta \tag{7}$$

and the boundary Γ_k is given, in turn, when F_{k-1} is known, by the following differential equation:

$$F_{k-1}((\rho-1)/\xi) + \xi^{-1}[1+\xi-\rho/2]\,F_{k-1}'((\rho-1)/\xi) = 0 \tag{8}$$

The initial data for relations (7) and (8) have the form

$$F_k(0) = 1, \ k = 0, 1, \ldots, \infty, \ F_0 \equiv 1, \ F_1(\xi^{-1/2}(1+\xi)^{-1/2}) = \xi/(1+\xi),$$
$$0 \leqslant \xi \leqslant \infty, \ \Gamma_1 = \{(\xi, \eta): \xi^{1/2}\eta = 1\} \tag{9}$$

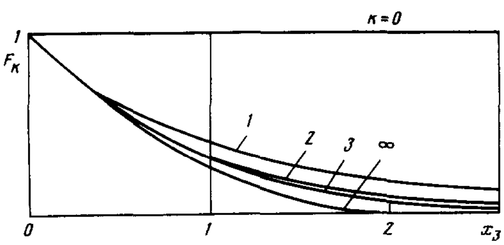

Fig.1

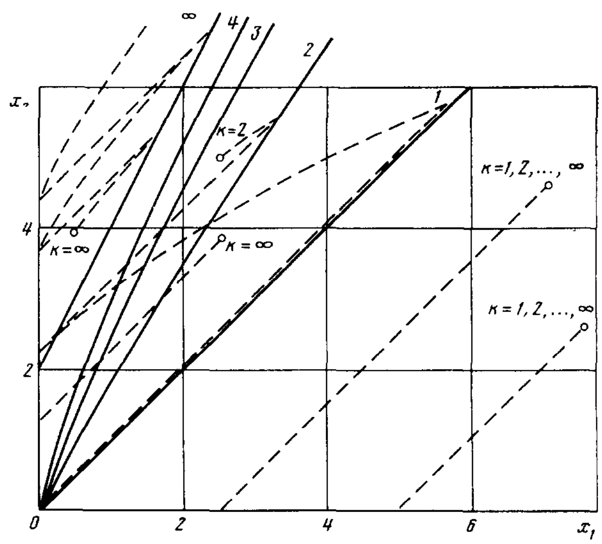

Fig.2

and $D_1 = \{(\xi, \eta): \xi^{1/2}\eta \leqslant 1\}$, $G_1 = \{(\xi, \eta): \xi^{1/2}\eta > 1\}$. Using relations (7)-(9) we can determine, one after the other, $F_1, \Gamma_2, F_2, \Gamma_3, \ldots, F_k, \Gamma_{k+1}, \ldots$. After this the regions D_k, G_k become known. The quantities S_k are then determined by the relations (6) and the quantities u_k are given by the formulas

$$u_k = -qz/|z|, \quad (z, q, \tau) \in D_1$$
$$u_k = -z/\tau^{1/2}, \quad (z, q, \tau) \in D_k/D_1, \quad k = 1, \ldots, N$$

Let us introduce the scalar variables y_1, y_2, y_3 using the relations

$$y_1 = \xi^{-1}, \quad y_2 = \eta\xi^{-1/2} - \xi^{-1}, \quad y_3 = \eta(1+\xi)^{-1/2} \tag{10}$$

Relations (7)-(9) written in the variables y_1, y_2 have the form

$$F_k(y_3) = \frac{F_{k-1}(y_2)}{1+y_1}, \quad (y_1, y_2) \in \Gamma_k, \quad y_3 = \frac{y_1+y_2}{(y_1+1)^{1/2}}, \quad k = 1, \ldots, N \tag{11}$$

$$y_1 = y_2 - 2 - 2F_{k-1}(y_2)/F'_{k-1}(y_2), \quad k = 2, \ldots, N \tag{12}$$

$$F_k(0) = 1, \quad k = 0, 1, \ldots, \infty, \quad F_0 \equiv 1, \quad F_1\left(\frac{y_1}{(y_1+1)^{1/2}}\right) = \frac{1}{1+y_1} \tag{13}$$

$$0 \leqslant y_1 \leqslant \infty, \ \Gamma_1 = \{(y_1, y_2) : y_2 \equiv 0, \ 0 \leqslant y_1 \leqslant \infty\}$$

Using relations (11)-(13), we constructed numerically two families of curves, namely, graphs of the functions $F_k(y_3)$ and the boundaries Γ_k. The results of computations in terms of the variables x_1, x_2, x_3, where $x_1 = y_1, x_2 = y_1 + y_2, x_3 = y_3$ are shown, for some values of the parameter k in Figs.1 and 2. The variables x_1, x_2 have an obvious meaning and are connected with the variables $|z|, \tau, q$ by the relations $x_1 = |z|/\tau, \ x_2 = q/\tau^{1/2}$. The function

$$F_\infty(x_3) = \begin{cases} \frac{1}{4}(2-x_3)^2, & 0 \leqslant x_3 < 2 \\ 0, & x_3 \geqslant 2 \end{cases}$$

is shown in Fig.1 and satisfies the limit relation $F_\infty = \lim F_k$ as $k \to \infty$.

The lines Γ_k are shown in Fig.2 by solid lines, and the numbers on the lines correspond to the value of the index k. The broken line $\Gamma_\infty = \{(x_1, x_2) : x_1 = 0, 0 \leqslant x_2 \leqslant 2\} \cup \{(x_1, x_2) : x_2 = 2x_1 + 2, 0 \leqslant x_1 \leqslant \infty\}$ in Fig.2 satisfies the limit relation $\Gamma_\infty = \lim \Gamma_k$ as $k \to \infty$. The points of the set $M = \{(x_1, x_2): x_1 > 0 \cup x_2 \geqslant 0\}$ above the curves Γ_k in Fig.2 form the regions G_k, and the sets D_k consist of the points of the set M below the curves Γ_k and include Γ_k. The dashed lines in Fig.2 show, for some values of the parameter k, the trajectories of motion of the point (x_1, x_2) obtained as a result of the application of the correction algorithm described above, under the worst effect of the noise. The initial points of these trajectories are indicated by circles and the corresponding values of the parameter k are given next to them.

The optimal synthesis for the case $k = \infty$ is represented by the curve Γ_∞, which defines D_∞ and G_∞. The special feature of the algorithm in this case consists of the fact that after the specified impulse one and the same line Γ_∞ is used as the signal curve. When the initial values of the parameters x_1, x_2, k satisfy the relations $x_2 \geqslant x_1 + 2, k = \infty$, the correction algorithm will differ from the algorithm obtained by passage to the limit (as $N \to \infty$, /1/) in the programmed problem. The discrepancy can be explained by the excess of correction resource Q at which complete correction $(J = 0)$ can be achieved using a number of different methods.

Thus the solution of synthesis problem (5) in the case when the number N of instances at which the impulses are applied is specified, is determined by the values of two continuous parameters and the index k. The scalar quantities x_1, x_2 can be used as the continuous parameters for problem (5). The synthesis for problem (5) was carried out by calculating the first instant t_1^* when the impulse is applied as the function ϑ of the parameter $|z^\circ|, N, Q, t_0$. It was also assumed that $z^\circ = 0, T = 1$ and the value of the function ϑ and the quantity Q were taken as two parameters of the game, determining the solution of the problem of synthesis for a given value of N. We can find from relations (5)-(7) and (10) that the functions $\vartheta_N(Q)$, shown in /1/ can be obtained from the curves in Fig.2 using the transformation

$$\vartheta = 1 - (1 + x_1)^{-1/2}, \ Q = x_2(1 + x_1)^{-1/2}/2 \tag{14}$$

Transforming the curve Γ_∞ by means of (14), we can obtain the curve $\vartheta_\infty(Q)$ which represents the limit position of the curves $\vartheta_N(Q)$ as $N \to \infty$. The curve $\vartheta_\infty(Q)$ is given by the relation

$$\vartheta_\infty(Q) = \begin{cases} 0, & 0 \leqslant Q \leqslant 1 \\ 1 - 1/Q, & 1 < Q \leqslant \infty \end{cases}$$

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